Image Reconstruction from Nonuniform Fourier Data

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Abstract

In many scientific frameworks (e.g., radio and high energy astronomy, medical imaging) the data at one's disposal are encoded in the form of sparse and nonuniform samples of the desired unknown object's Fourier Transform. This work aims at reconstructing the object's image by acting straightly on the data without interpolation and re-sampling operations, which might affect the reconstruction's quality. In particular, we show that the minimization of the data discrepancy is equivalent to a deconvolution problem with a suitable kernel and we address its solution by means of a gradient projection method with an adaptive steplength parameter. Since the gradient of the objective function involves a convolution operator, the algorithm can be effectively implemented exploiting the Fast Fourier Transform. Further information can be found in [1].

Mathematical formulation

Unknown object:

$$[X_1, X_2] \times [Y_1, Y_2] \equiv \mathcal{D} \ni (x,y) \mapsto f(x,y) \in \mathbb{R}$$

Available data (also called **visibilities**): for k=1,…,N,

$$g_k \equiv (Af)_k = \int_{\mathcal{D}} f(x,y) e^{2\pi i (u_k x + v_k y)} dx dy$$

Adjoint operator (back – projection):

$$(A^* c)(x,y) = \sum_{k=1}^{N} c_k e^{-2\pi i (u_k x + v_k y)}, \qquad (x,y) \in \mathcal{D}$$

Minimization problem (due to the ill – posedness of Af = g):

$$\min_{\substack{f \in L^2(\mathcal{D},\mathbb{R}) \\ f \geq 0}} J(f) \equiv \tfrac{1}{2} \|Af - g\|^2_{\mathbb{C}^N}$$

Gradient of the objective function:

$$\nabla J(f) = A^*Af - A^*g = D * f - f_d$$

"dirty beam" convolution "dirty map"

$$D(x,y) = \sum_{k=1}^{N} e^{-2\pi i (u_k x + v_k y)} \qquad f_d(x,y) = \sum_{k=1}^{N} g_k e^{-2\pi i (u_k x + v_k y)}$$

(fast calculation through FFT)

Discretization:

$$x_j = X_1 + (j-1)\Delta x, \ y_h = Y_1 + (h-1)\Delta y, \quad j,h = 1,\dots,n$$

$$g_k = (Af)_k \approx \sum_{j,h=1}^{n} f(x_j, y_h)\Delta x \Delta y \, e^{2\pi i (u_k x_j + v_k y_h)}, \ k = 1,\dots,N$$

$$=: f_{jh}$$

Gradient Projection Method (GPM)

Let $\mathcal{C}$ be the set of all nonnegative images.
Choose the starting point $f^{(0)} \in \mathcal{C}$, set the parameters $\beta, \theta \in (0,1)$, $0 < \alpha_{min} < \alpha_{max}$.

FOR $k = 0,1,2,\dots$ DO THE FOLLOWING STEPS:

STEP 1. Choose the parameter $\alpha_k \in [\alpha_{\min}, \alpha_{\max}]$.

STEP 2. Projection: $y^{(k)} = P_{\mathcal{C}}(f^{(k)} - \alpha_k \nabla J(f^{(k)}))$.

STEP 3. Descent direction: $d^{(k)} = y^{(k)} - f^{(k)}$.

STEP 4. Set $\lambda_k = 1$.

STEP 5. Backtracking loop:
- let $J_{new} = J(f^{(k)} + \lambda_k d^{(k)})$;
- IF $J_{new} <= J(f^{(k)}) + \beta \lambda_k \nabla J(f^{(k)})^T d^{(k)}$ THEN
  - go to Step 6;
- ELSE
  - set $\lambda_k = \theta \lambda_k$ and go to Step 5.
- ENDIF

STEP 6. Set $f^{(k+1)} = f^{(k)} + \lambda_k d^{(k)}$.

END

Adaptive alternation of the Barzilai – Borwein rules [2,3]

$$\alpha_k^{(1)} = \frac{{s^{(k-1)}}^T s^{(k-1)}}{{s^{(k-1)}}^T z^{(k-1)}} \quad ; \quad \alpha_k^{(2)} = \frac{{s^{(k-1)}}^T z^{(k-1)}}{{z^{(k-1)}}^T z^{(k-1)}}$$

$$s^{(k-1)} = f^{(k)} - f^{(k-1)} \ ; \ z^{(k-1)} = \nabla J(f^{(k)}) - \nabla J(f^{(k-1)})$$

IF $\alpha_k^{(2)} / \alpha_k^{(1)} <= \tau_k$ THEN

$\quad \alpha_k = \min\left\{\alpha_j^{(2)}, \ j = \max\{1, k - M_\alpha\}, \dots, k\right\}; \ \ \tau_{k+1} = \tau_k * 0.9;$

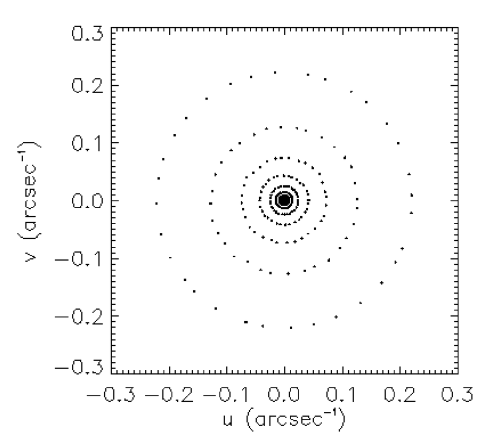
ELSE

$\quad \alpha_k = \alpha_k^{(1)}; \ \ \tau_{k+1} = \tau_k * 1.1;$

ENDIF $\qquad (M_\alpha \in \mathbb{Z}_+, \tau_1 \in (0,1))$

Numerical tests: the RHESSI mission

The Reuven Ramaty High Energy Solar Spectroscopic Imager (*RHESSI*) [4], launched by NASA on February 2002, produces images with the finest angular and spectral resolution ever achieved at hard X-ray and γ-ray energies. Such **imaging spectroscopy** provides a powerful tool with which to explore the underlying physics of particle acceleration and transport in solar flares.
RHESSI encodes spatial information through the temporal modulation of photon flux by a set of nine Rotating Modulation Collimators (RMCs) [5]. This information is rather straightforwardly converted to **visibilities**, which are 2D spatial Fourier components corresponding to spatial frequencies (u,v) lying on nine concentric circles.



Imaging from visibilities

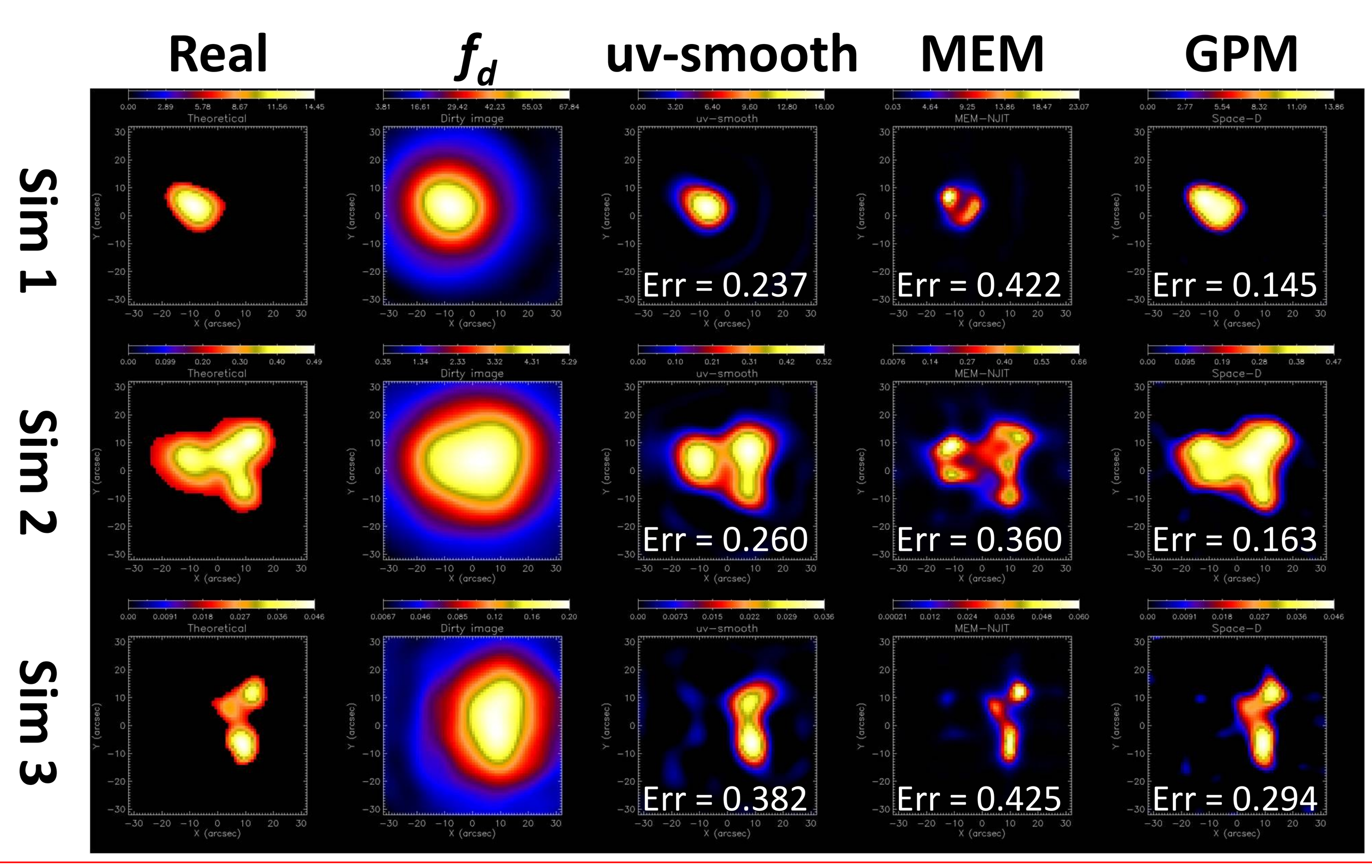
Algorithms available in RHESSI Solar SoftWare (SSW):

- uv-smooth [6]: interpolation + resampling + projected Gerchberg-Papoulis method
- MEM [7]: interpolation + resampling + maximum entropy method
- back projection, forward fit

Simulated datasets creation

- select a real dataset and reconstruct the related image with a method at will
- clean the image from artifacts (by zeroing all the pixels lower than a fixed threshold)
- Fourier Transform the resulting image to get the "perfect" visibilities
- corrupt the visibilities with realistic noise

Reconstructions



References

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