Image Regularization for Poisson Data

Alessandro Benfenati, Valeria Ruggiero

Department of Mathematics and Computer Science, University of Ferrara

Regularization Parameter: Three Models

In many imaging applications, the detected data $g \in \mathbb{R}^m$ is a realization of a Poisson multivalued random variable: $g \sim \mathcal{P}oisson(Hx + b)$, where $H \in \mathbb{R}^{m \times n}$ is the imaging matrix, $b \geq 0$ is the constant background emission and $x \in \mathbb{R}^n$ is the incoming signal. In a Bayesian framework, the true image x^* is a realization of a m.r.v. with pdf $\exp(-\beta\varphi(x))$; an approximation x_β is obtained by maximizing the *a posteriori probability* $P(x_\beta|g)$, or by solving the equivalent minimization problem

$$x_\beta = \arg\min_{x \geq 0} D_{\mathcal{KL}}(g; Hx + b) + \beta\varphi(x)$$

where $D_{\mathcal{KL}}$ is the generalized Kullback–Leibler function, φ is a regularization function and $\beta > 0$ is the regularization parameter. **The estimation of the optimal value for β is very hard in presence of Poisson noise**. We propose 3 different approaches:

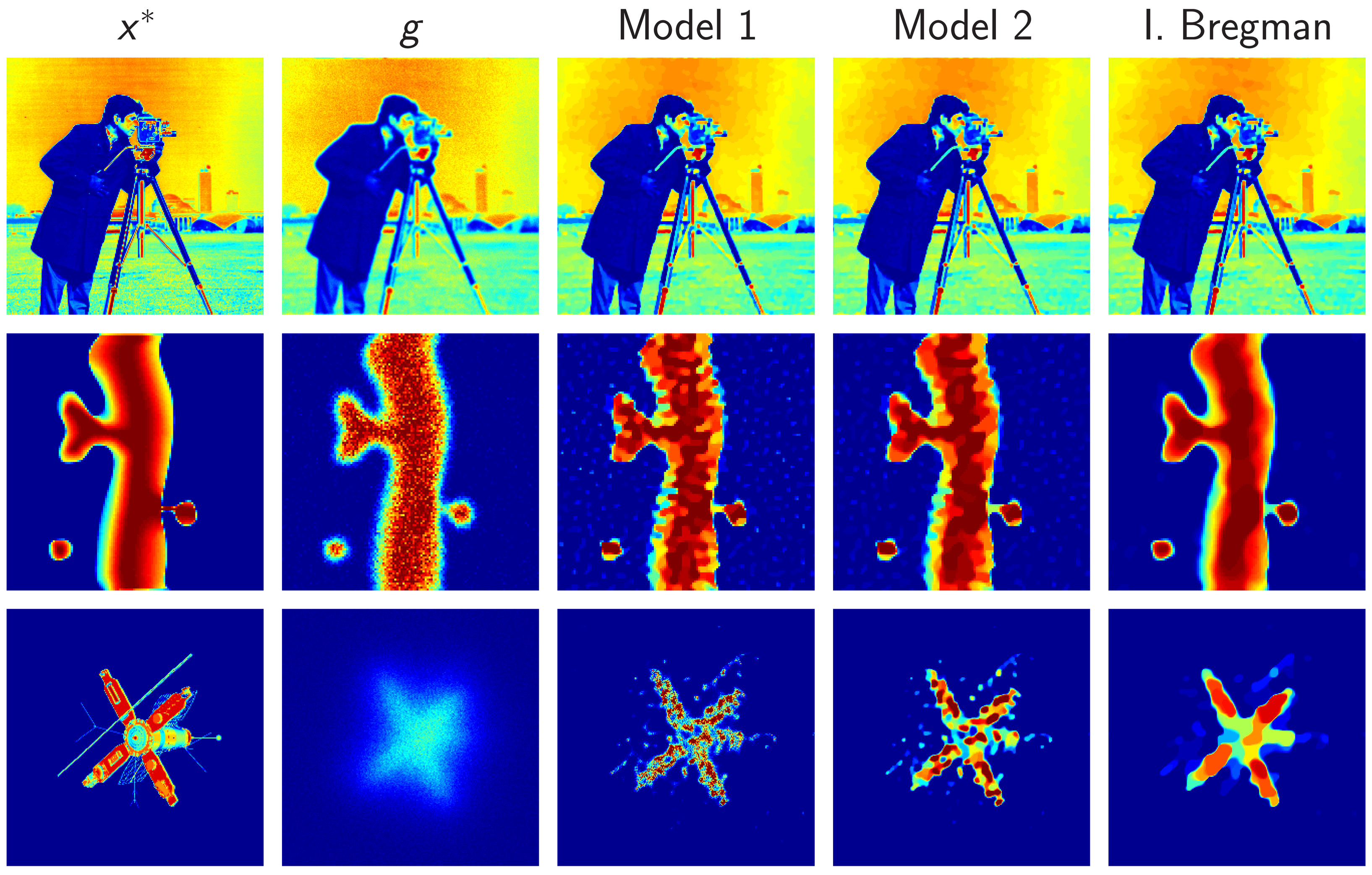
Discrepancy Model or **Model 1**
Based on Lemma 1: β is estimated by finding the root of a *discrepancy* equation

Constrained Model or **Model 2**
β is estimated by adopting a *constrained* approach (based on Lemma 1).

Inexact Bregman procedure
It allows to use an overestimation of the optimal value β_{opt} of the regularization parameter.

The first two models do not always provide reliable results in presence of low counts images. The third approach enables to obtain very promising results in case of low counts images and High Dynamic Range astronomical images.

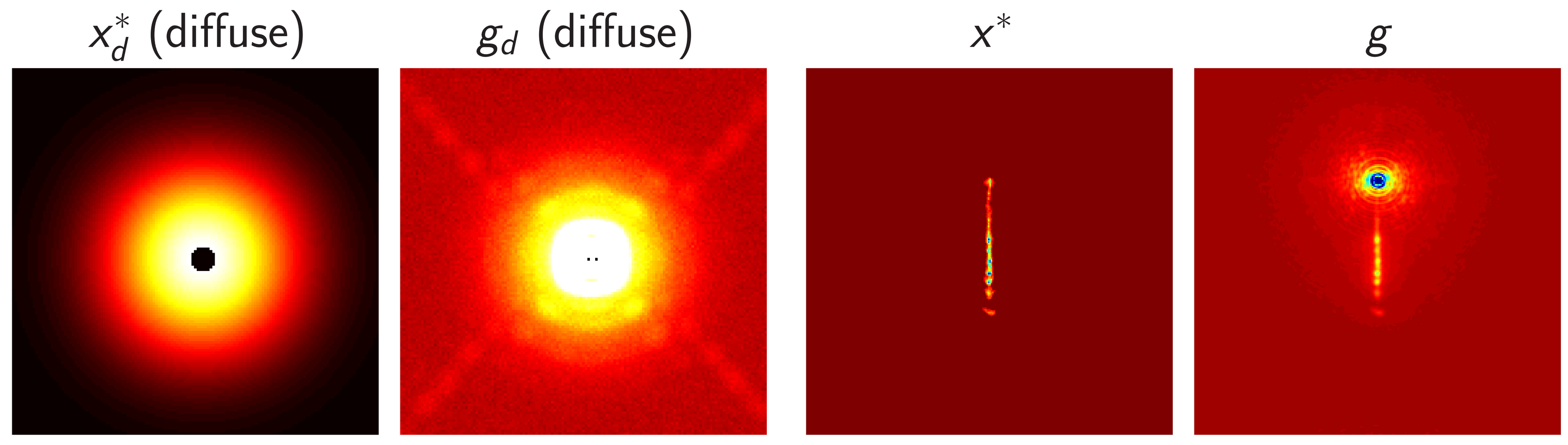
Comparison Between the three models



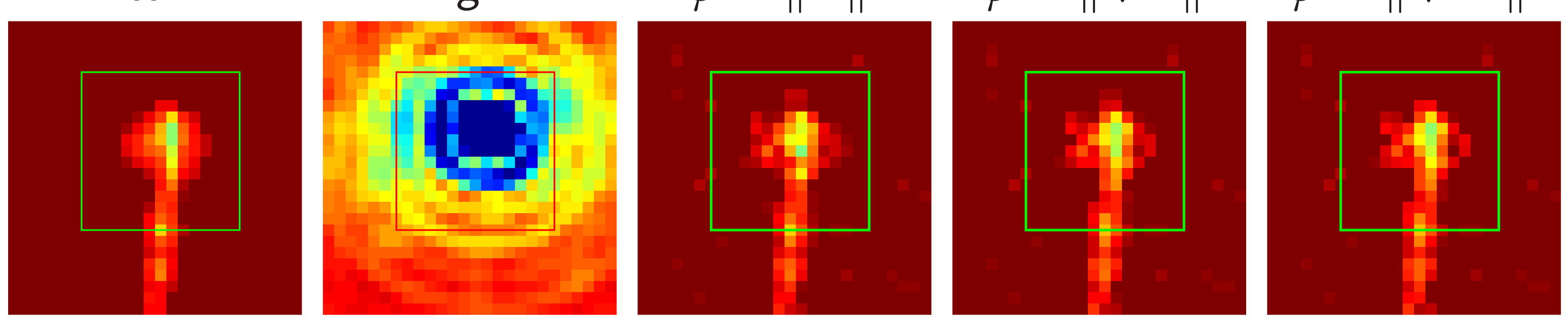
Test problem	Model	k_{ext}	k_{tot}	β_k	ρ
cameraman	✓Model 1	8	815	$6.689 \cdot 10^{-3}$	$8.562 \cdot 10^{-2}$
	✓Model 2	451		$6.699 \cdot 10^{-3}$	$8.535 \cdot 10^{-2}$
	✓I. Bregman	6	3906		$8.730 \cdot 10^{-2}$
micro	✗Model 1	11	1458	$3.374 \cdot 10^{-3}$	$1.658 \cdot 10^{-1}$
	✗Model 2	*5000		$7.637 \cdot 10^{-3}$	$1.294 \cdot 10^{-1}$
	✓I. Bregman	9	4615		$8.370 \cdot 10^{-2}$
spacecraft	✗Model 1	41	3731	$1.000 \cdot 10^{-41}$	$1.000 \cdot 10^{0}$
	✗Model 2	*5000		$1.501 \cdot 10^{-4}$	$5.098 \cdot 10^{-1}$
	✓I. Bregman	9	27480		$3.780 \cdot 10^{-2}$

Numerical results. k_{ext} is the number of external iterations, k_{tot} is the total number of internal iterations, β estimate, and relative reconstruction error. For I. Bregman procedure, $\beta = 10\beta_{opt}$. In case of low counts images (micro and spacecraft), Model 1&2 can not reach β_{opt} (0.0477 and 0.00163, respectively).

Astronomical case



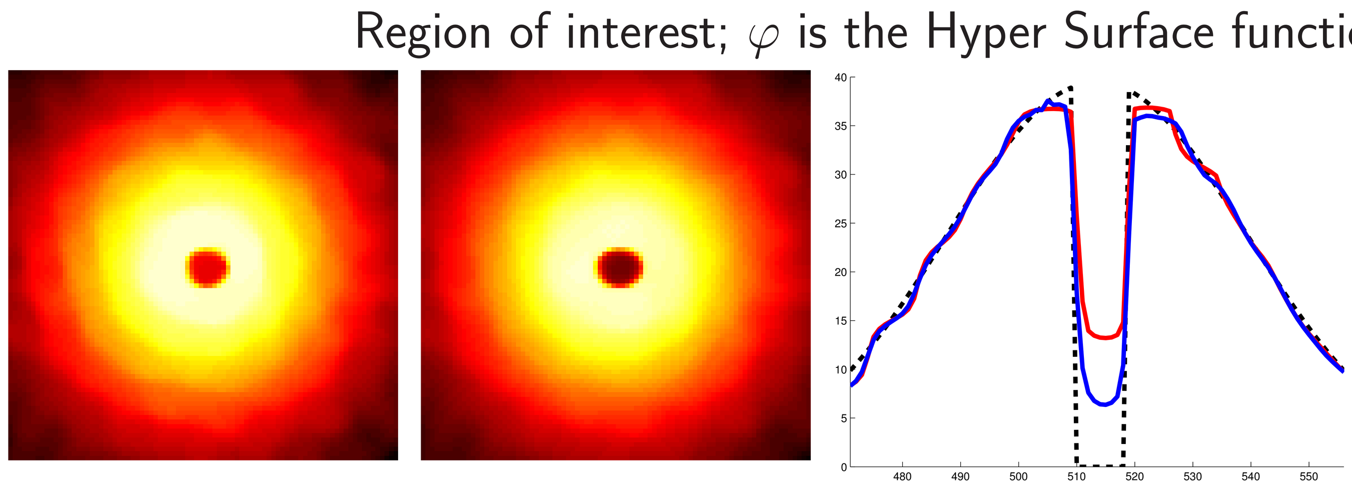
The signal is divided into the point–source component x_p and the diffuse one x_d: $x = x_p + x_d$. φ acts only on the diffuse component.



Detail; ρ_w refers to the square $\rho_w = 0.32$ $\rho_w = 0.33$ $\rho_w = 0.25$

Comparison between an optimal–tuned method (Scaled Gradient Projection, SGP) with the I. Bregman procedure ($\beta = 10\beta_{opt}$).

Region of interest; φ is the Hyper Surface functional



SGP, β_{opt} I. Bregman

513rd profile line
Blue: I. Breg. Proc.
Red: SGP rec.
Black: x^*

The Three Models

Lemma. *Let Y_λ a Poisson random variable with expected value λ and consider the following function* $F(Y_\lambda) = 2\left\{Y_\lambda \log\left(\frac{Y_\lambda}{\lambda}\right) + \lambda - Y_\lambda\right\}$. *Then, for large λ the following asymptotic estimate of the expected value $E[F(Y_\lambda)]$ holds true:*

$$E[F(Y_\lambda)] = 1 + \mathcal{O}\left(\frac{1}{\lambda}\right)$$

The **Discrepancy Model** (Model 1) consists in solving

$$\mathcal{D}_H(x_\beta; g) = \eta \sim 1$$

with $\mathcal{D}_H(x_\beta; g) \equiv 2m^{-1}D_{\mathcal{KL}}(g; Hx + b)$ (Bertero et al, 2010).

The **Constrained Model** (Model 2) consists in solving the problem

$$\min_{x \geq 0} \varphi(x) \text{ subject to } \mathcal{D}_H(x; g) \leq \eta \sim 1$$

(Teuber et al, 2013). Under suitable assumptions on φ, Model 1 and Model 2 provide the same parameter estimation when $E[Hx^* + b]$ is large.

The **Inexact Bregman Procedure** is based on the inexact Bregman distance

$$\Delta_\varepsilon^\xi \varphi(x, y) = \varphi(x) - \varphi(y) - \langle \xi, x - y \rangle + \varepsilon$$

with $\xi \in \partial_\varepsilon\varphi(y)$; providing μ_k and ν_k s.t $\sum_{i=1}^\infty \mu_i < \infty$ and $\sum_{i=1}^\infty i\nu_i < \infty$ the procedure consists in

For $k = 0, 1, 2, \ldots$ do

$$x^{k+1} \sim \arg\min_{x \geq 0} D_{\mathcal{KL}}(g, Hx + b) + \beta\Delta_{\varepsilon_k}^{\xi^k}\varphi(x^k, x) \quad (1)$$

s.t.

$$\|\eta_{k+1}\| \leq \mu_{k+1} \text{ and } \varepsilon_{k+1} \leq \nu_{k+1}$$

where η_{k+1} is an ε_{k+1}–subgradient of the objective function in (1). This procedure allows to use an overestimation of the optimal value β_{opt} of the parameter β. By early stopping, this procedure has a regularization behaviour (Benfenati et al, 2013).