Image Regularization for Poisson Data  
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**Regularization Parameter: Three Models**

In many imaging applications, the detected data \( g \in \mathbb{R}^m \) is a realization of a Poisson multivalued random variable: \( g \sim \text{Poisson}(Hx + b) \), where \( H \in \mathbb{R}^{m \times n} \) is the imaging matrix, \( b \geq 0 \) is the constant background emission and \( x \in \mathbb{R}^n \) is the incoming signal. In a Bayesian framework, the true image \( x^* \) is a realization of a m.r.v. with pdf \( \exp(-\varphi(x)) \); an approximation \( x_\beta \) is obtained by maximizing the a posteriori probability \( P(x|g) \), or by solving the equivalent minimization problem

\[
x_\beta = \arg \min_{x \in \mathbb{R}^n} D_{KL}(g; Hx + b) + \beta \varphi(x)
\]

where \( D_{KL} \) is the generalized Kullback-Leibler function, \( \varphi \) is a regularization function and \( \beta > 0 \) is the regularization parameter. The estimation of the optimal value for \( \beta \) is very hard in presence of Poisson noise. We propose 3 different approaches:

- **Discrepancy Model or Model 1**
  Based on Lemma 1: \( \beta \) is estimated by finding the root of a discrepancy equation

- **Constrained Model or Model 2**
  \( \beta \) is estimated by adopting a constrained approach (based on Lemma 1).

- **Inexact Bregman procedure**
  It allows to use an overestimation of the optimal value \( \beta_{opt} \) of the regularization parameter.

The first two models do not always provide reliable results in presence of low counts images. The third approach enables to obtain very promising results in case of low counts images and High Dynamic Range astronomical images.

**Comparison Between the three models**

![Comparison Between the three models](image)

**Astronomical case**

- **Binary stars**
- **Young Stellar Objects**

The signal is divided into the point–source component \( x_p \) and the diffuse one \( x_d \): \( x = x_p + x_d \). \( \varphi \) acts only on the diffuse component.

![Astronomical case](image)

**The Three Models**

**Lemma.** Let \( Y_\lambda \) a Poisson random variable with expected value \( \lambda \) and consider the following function \( F(Y_\lambda) = 2 \left\{ Y_\lambda \log \left( \frac{\lambda}{Y_\lambda} \right) + \lambda - Y_\lambda \right\} \). Then, for large \( \lambda \) the following asymptotic estimate of the expected value \( E[F(Y_\lambda)] \) holds true:

\[
E[F(Y_\lambda)] = 1 + O \left( \frac{1}{\lambda} \right)
\]

The **Discrepancy Model** (Model 1) consists in solving

\[
D_{KL}(x; g) = \eta \sim 1
\]

with \( D_{KL}(x; g) \equiv 2m^{-1}D_{KL}(g; Hx + b) \) (Bertero et al, 2010).

The **Constrained Model** (Model 2) consists in solving the problem

\[
\min_{x \in \mathbb{R}^n} \varphi(x) \text{ subject to } D_{KL}(x; g) \leq \eta \sim 1
\]

(Teuber et al, 2013). Under suitable assumptions on \( \varphi \), Model 1 and Model 2 provide the same parameter estimation when \( E[Hx^* + b] \) is large.

The **Inexact Bregman procedure** is based on the inexact Bregman distance

\[
\Delta^k_{\beta}(x, y) = \varphi(x) - \varphi(y) - \langle \xi, x - y \rangle + \epsilon
\]

with \( \xi \in \partial \varphi(y) \); providing \( \mu_k \) and \( \nu_k \) s.t \( \sum_{i=1}^{k} \mu_i < \infty \) and \( \sum_{i=1}^{k} \nu_i < \infty \) the procedure consists in

For \( k = 0, 1, 2, \ldots \)

\[
x^{k+1} = \arg \min_{x \in \mathbb{R}^n} D_{KL}(g; Hx + b) + \beta \Delta^k_{\beta}(x^k, x)
\]

s.t.

\[
\|\eta_{k+1}\| \leq \mu_k \text{ and } \varepsilon_{k+1} \leq \nu_{k+1}
\]

where \( \eta_{k+1} \) is an \( \varepsilon_{k+1} \)-subgradient of the objective function in (1).

This procedure allows to use an overestimation of the optimal value \( \beta_{opt} \) of the parameter \( \beta \). By early stopping, this procedure has a regularization behaviour (Benfenati et al, 2013).

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