Introduction to 2d Computed Tomography

This work addresses the problem of 2d imaging reconstruction from few tomographic real data. Placing an object between an X-ray source and a detector, one of its slices is crossed by an X-ray fan-beam and we want to reconstruct its inner part. In modern Computed Tomography (CT) several projections are executed but it could be important to reduce the number of scans, avoiding to lose too much in quality. Here we compare how two different and efficient algorithms behave changing the sparsity of the views and working on a dataset obtained from a real tomographic experiment.

In our case, the CT image formation process is modelled as an underdetermined linear system

\[ Ax = b \]

and the reconstructed image \( x \) is the solution of the regularized problem:

\[ \min_{x \in \mathbb{R}^n} f(x) = \frac{1}{2} \|Ax - b\|^2 + \lambda \text{TV}(x) \]

- \( x \) is the slice of interest, discretized in \( N := N_x \times N_y \) pixels;
- \( b \) is the collection of all the noisy projection measurements, recorded by a linear detector (made of \( n_{\text{cells}} \) elements) from \( n_{\theta} \) angles taken on a circular trajectory (\( n := n_{\text{cells}} \times n_{\theta} \));
- \( A \) is a matrix of size \( n \times N \), representing how the projector works on the detector;
- \( \text{TV}(x) := \sum_{j=1}^{N} (\|D_j x\|^2 + \beta^2)^{\frac{1}{2}} \) is the differentiable discrete Total Variation of \( x \) (with \( D_j x \in \mathbb{R}^2 \) 2d-discrete gradient of pixel \( x_j \)) and \( \lambda > 0 \) is the regularization parameter.

Due to the incompleteness of the projection data, \( n \ll N \) and the image reconstruction from real data becomes a challenging inverse problem.

**Scaled Gradient Projection (SGP) algorithm**

The constrained problem \( \min_{x \geq 0} f(x) \) is solved with:

**SGP algorithm**

Initialize: \( x_0 \geq 0, \delta, \sigma \in (0,1), 0 \leq \alpha_{\text{min}} \leq \alpha_0 \leq \alpha_{\text{max}}, D_0 \in \mathbb{R}^p \);
while (convergence)

\[ d_k = \nu_D(x_k - \alpha D_k \nabla f(x_k)) - x_k; \quad (\text{scaled gradient projection step}) \]
\[ \lambda_k = 1; \]
while \( f(x_k + \lambda_k d_k) > f(x_k) + \sigma \lambda_k \|\nabla f(x_k)\|_2 d_k \)
\[ \lambda_k = \delta \lambda_k; \quad (\text{backtracking step}) \]

\[ x_{k+1} = x_k + \lambda_k d_k; \]
define the diagonal scaling matrix \( D_{k+1} \in \mathbb{R}^p\{\text{scaling updating rule}\} \)

\[ \alpha_{k+1} \in [\alpha_{\text{min}}, \alpha_{\text{max}}]; \quad (\text{step-length updating rule}) \]

\[ k = k + 1; \]
end

**Fixed Point (FP) algorithm**

Considering \( L_k = L(x_k) \) where \( L(x) \) is an operator such that \( L(x_k) x_k = \nabla \text{TV}(x_k) \), the optimization problem is solved with:

**FP algorithm**

Initialize: \( x_0 \)
while (convergence)

\[ g_k = \nabla f(x_k); \]
\[ H_k = A^T A + \lambda L(x_k); \quad (\text{Hessian approximation}) \]
\[ s_k \] such that \( H_k s_k = -g_k; \quad (\text{resolution of an inner inverse problem, with CG method}) \]

\[ x_{k+1} = x_k + s_k; \]
\[ k = k + 1; \]
end

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**Numerical results**

- \( N_x = N_y = 256 \) (hence \( N = 65536 \))
- angular range \([0, 2\pi]\)
- \( n_{\theta} = 120 \) or \( n_{\theta} = 20 \)
- \( n_{\text{cells}} = 429 \) (hence \( n = 51480 \) or \( n = 8580 \))
- data and image accuracies \( \approx 280 \mu m \)

**Algorithm parameters:**

- \( \lambda = 10^{-2} \) for SGP, \( \lambda = 10^{-2} \) for FP
- \( \beta = 10^{-3} \)
- \( \text{maxiter}_{\text{cg}} = 10 \)

**Dataset:**

- the object of interest is a central slice of a piece of lotus root filled with
  - one circular chalk
  - square pieces of ceramics
  - a section of a pencil
  - fragments of matches

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**Results from 120 projections**

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<th>Results from 20 projections</th>
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