A scaled $\varepsilon$–subgradient method
Alessandro Benfenati‡, Silvia Bonettini‡, Valeria Ruggiero‡

Department of Mathematics and Computer Science, University of Ferrara, ‡Université Paris–Est LIGM

The optimization problem reads as
\[
\min_{x \in \mathbb{R}^n} f(x) + \Phi(x)
\]

$\Phi$ convex, proper, l.s.c function, $\text{dom}(\Phi) \subset \text{dom}(f)$

Assume that both the $\varepsilon$–subgradients of $f$ and $\Phi$ and the eigenvalues of $D_k$ are bounded. Let $\varepsilon_k = \inf_{x \in \mathbb{R}^n} (f(x) + \Phi(x))$ and define $X^*$ as the set of solutions; under the assumptions on (1) one has

\[
\lim_{k \to \infty} \varepsilon_k = 0, \quad \sum_{k=0}^{\infty} \varepsilon_k = \infty, \quad \sum_{k=0}^{\infty} \epsilon_k \alpha_k < \infty, \quad \sum_{k=0}^{\infty} \alpha_k < \infty
\]

If $X^* = \emptyset$, $\varepsilon_k$ is unbounded.

The convergence rate is quite pessimistic $\left(\sum_{k=0}^{\infty} \alpha_k^{-1}\right)$, but the numerical experience shows that the actual performance of the scaled method overcomes the non scaled version.

A choice for $\alpha_k$

**Dynamic rule**

\[
\alpha_k = \frac{f(x^k) - f^*}{\|u^k\|^2} \quad \text{or} \quad \alpha_k = \frac{f(x^k) - f^*}{\max\{1, \|u^k\|^2\}}
\]

Assumption: $\varepsilon$-subgradients of $f$ and $\Phi$ bounded.

Inspired by the Polyak rule, $f_k$ is an estimation of $f^*$: a level algorithm (Goffin 99) is employed to obtain such an estimation.

The Scaled Primal Dual Hybrid Algorithm (SPDHG) reads as

\[
y^{k+1} = \text{prox}_{\tau_k \phi} (y^k + \gamma_k A^* x^k)
\]

\[
u^{k+1} = \text{prox}_{\gamma_k \psi} (u^k - \frac{\tau_k}{\gamma_k} A x^{k+1})
\]

\[
x^{k+1} = \text{prox}_{\tau_k \phi} (x^k - \gamma_k D_k u^{k+1})
\]

Assume that $d^* = \nabla f(x^*), A^* y^{k+1} \in \partial \Phi(x^*)$ and the eigenvalues of $D_k$ are bounded.

\[
\alpha_k = \mathcal{O}\left(k^{-p}\right), \quad \tau_k = \mathcal{O}\left(k^{-q}\right)
\]

If $\text{diam}(\text{dom}(f_k)) < \infty$ then

\[
\lim_{k \to \infty} \inf_{x} f(x^k) + \Phi(x^k) = f^*
\]

If the set of the solutions $X^* = \emptyset$, then

\[
\lim_{k \to \infty} f(x^k) + \Phi(x^k) = f^* = f(x^*)
\]

### Application: Image Restoration with Poisson Noise

Let consider a blurred image affected by Poisson noise: the aim is to restore the image by solving

\[
\min_{x \in \mathbb{R}^n} f(x) + \Phi(x) \equiv f_0(x) + f_1(Ax) + \Phi(x)
\]

\[
f_0(x) = \sum_{i=1}^{n} g_i \| (Hx)_i + b - g_i \|_{L^1}
\]

\[
f_1(Ax) = \beta \sum_{i=1}^{n} \| A_i x_i \|_1, \quad A_i \in \mathbb{R}^{2 \times n} (\text{Total Variation})
\]

The scaled primal dual hybrid algorithm (SPDHG) reads as

\[
y^{k+1} = \text{prox}_{\tau_k \phi} (y^k + \gamma_k A^* x^k)
\]

\[
u^{k+1} = \text{prox}_{\gamma_k \psi} (u^k - \frac{\tau_k}{\gamma_k} A x^{k+1})
\]

\[
x^{k+1} = \text{prox}_{\tau_k \phi} (x^k - \gamma_k D_k u^{k+1})
\]

Assume that $d^* = \nabla f(x^*), A^* y^{k+1} \in \partial \Phi(x^*)$ and the eigenvalues of $D_k$ are bounded.

\[
\alpha_k = \mathcal{O}\left(k^{-p}\right), \quad \tau_k = \mathcal{O}\left(k^{-q}\right)
\]

If $\text{diam}(\text{dom}(f_k)) < \infty$ then

\[
\lim_{k \to \infty} \inf_{x} f(x^k) + \Phi(x^k) = f^*
\]

If the set of the solutions $X^* = \emptyset$, then

\[
\lim_{k \to \infty} f(x^k) + \Phi(x^k) = f^* = f(x^*)
\]

### Convergence Results

All the technical details and complete references are available in
S. Bonettini, A. Benfenati, and V. Ruggiero, Scaling Techniques for $\varepsilon$-Subgradient Methods, SIAM Journal on Optimization 2016 26:3, 1741-1772.

Optimiziation Techniques For Inverse Problems III - 20 September 2016