Steplength selections in gradient projection methods with applications to image deblurring

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Abstract
In the framework of gradient projection-based approaches, steplength selection techniques are very crucial for the convergence of the method. In the context of constrained optimization, we propose modified versions of the well-known Barzilai-Borwein rules (and their extensions), highlighting their feature of capturing second-order information in a low cost way, as in the unconstrained case ([1]). The practical effectiveness of the proposed strategy has been shown on random large scale box-constrained quadratic problems, on some well-known non-convex problems, and on image deblurring applications.

Mathematical Framework
Let consider the following box-constrained quadratic programming (BQP) problem
\[
\min_{x \in S} f(x) = \frac{1}{2} x^T A x - b^T x + c
\]
where \( A \in \mathbb{R}^{n \times n} \) is symmetric positive definite, \( b \in \mathbb{R}^n \) and \( c \in \mathbb{R} \).
Let denote \( g(x) = \nabla f(x) \). We solve problem (1) by means of the gradient projection (GP) algorithm combined with a line-search strategy along the feasible direction. The main step of the GP method is described in Algorithm 1.

Algorithm 1: GP method for box-constrained quadratic programs

Initialization: choose \( x^0 \in \mathbb{R}^n, f(x^0) \leq 0 \) for \( i = 0, 1, \ldots \) do
\[
\begin{align*}
& x^i = x^i - \alpha^i (g(x^i))^\top g(x^i) \\
& \text{if } f(x^i) < 0 \text{ then } \text{go to } x^{i+1} \text{ else } \text{stop}
\end{align*}
\]
end

Steplength selection strategies
Unconstrained case. The standard Barzilai–Borwein (BB) rules [1] are obtained by imposing
\[
a_{BB} = \min_{\alpha \in \mathbb{R}} \{ f(x) - f(x^i) \mid \alpha \} = \min_{\alpha \in \mathbb{R}} \{ f(x) - f(x^i) \mid \alpha \}
\]
and
\[
b_{BB} = \min_{\alpha \in \mathbb{R}} \{ g(x)^\top g(x) \mid \alpha \} = \min_{\alpha \in \mathbb{R}} \{ g(x)^\top g(x) \mid \alpha \}
\]
where \( f(x) = \frac{1}{2} x^T A x - b^T x + c \) and \( g(x) = \nabla f(x) \).

Well-known improvements of the BB rules are the strategies Alternate Barzilai–Borwein (ABB) [2] and its modification ABB\(m\) [4].

The next theorem states that steplengths (4)-(6) are the reciprocal of the Rayleigh quotients of \( A_{BB} \).

Theorem 1
If \( A \) in (1) is a symmetric positive definite matrix, we have
\[
\lambda_{\min}(A_{BB}) \leq \min_{\alpha \in \mathbb{R}} \{ f(x) - f(x^i) \mid \alpha \} \leq \lambda_{\max}(A_{BB})
\]
\[
\lambda_{\min}(A_{BB}) \leq \min_{\alpha \in \mathbb{R}} \{ g(x)^\top g(x) \mid \alpha \} \leq \lambda_{\max}(A_{BB})
\]
As a consequence of the previous results, we can consider a modified ABB\(m\) scheme consisting in the alternation between BB1 and BB2.

Application to image deblurring
Data affected by Gaussian noise
The test problems are generated by convolving the original \( 256 \times 256 \) images (named Nebula, and Spacecraft) with a point spread function (simulated ground-based telescope http://www.maths.emosyrd.edu.au/Nagy/RestoreTools/index.html), and perturbing the results with additive white Gaussian noise with variance \( 1 \) and zero background noise [5]. The corresponding constrained minimization problem is a least squares problem with non-negative constraints of the form:
\[
\min_{x \in \mathbb{R}^n} \| Ax - y \|_2^2
\]
where \( y \in \mathbb{R}^m \) is the non-negative observed data, \( A \in \mathbb{R}^{m \times n} \) is the imaging matrix, and \( x \in \mathbb{R}^n \) is the image to recover. Hereafter, we denote by RRE the relative reconstruction error, and by Ad-ABB\(m\) and Ad-MABB\(m\), respectively, the alternate approaches when the threshold \( r \) is variable instead of being a constant parameter.

For forthcoming research
✓ Generalization of the strategy to other feasible regions
✓ Steplengths selection rule based on Ritz-like values for constrained problems
✓ Analysis of the behaviour of modified steplengths rules in presence of a variable metric

References