An image reconstruction method from Fourier data with uncertainties on the spatial frequencies

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Abstract

In this work we develop a new optimization algorithm for image reconstruction problems from Fourier data with uncertainties on the spatial frequencies corresponding to the measured data. By considering such dependency on the frequencies as a further unknown, we obtain a so-called semi-blind deconvolution. Both the image and the spatial frequencies are obtained as solutions of a reformulated constrained optimization problem, approached by an alternating scheme. Numerical tests on simulated data, based on the imaging hardware of the NASA RHESSI satellite, show that the proposed approach provides some improvements in the reconstruction.

Mathematical formulation

Fourier-based image restoration can be modeled as an ill-posed inverse problem

$$\mathbf{g} = \mathcal{A}(\boldsymbol{\theta}, \boldsymbol{\rho})\mathbf{f} + \boldsymbol{\eta}$$

where

- $\mathbf{g} \in \mathbb{C}^N$ is the available complex data (called *visibilities*)
- $\mathbf{f} \in \mathbb{R}^{n^2}$ is the unknown true image
- $(\boldsymbol{\theta}, \boldsymbol{\rho}) \in \mathbb{R}^N$ are the unknown spatial frequencies in polar coordinates
- $\mathcal{A}$ is the discrete Fourier transform, $k = 1, \ldots, N$

$$(\mathcal{A}(\boldsymbol{\theta}, \boldsymbol{\rho})\mathbf{f})_k = \sum_{j,h=1}^{n} \mathbf{f}_{jh} \exp\left(2\pi i (x_j \cos(\theta_k) + y_h \sin(\theta_k))\right),$$

- $\boldsymbol{\eta}$ models the noise

The vector of unknowns $(\mathbf{f}, \boldsymbol{\theta}, \boldsymbol{\rho})$ can be obtained as the solution of the constrained optimization problem [2]

$$\min_{\substack{\mathbf{f} \geq 0 \\ \boldsymbol{\theta}^{min} \leq \boldsymbol{\theta} \leq \boldsymbol{\theta}^{max} \\ \boldsymbol{\rho}^{min} \leq \boldsymbol{\rho} \leq \boldsymbol{\rho}^{max}}} \mathcal{F}(\mathbf{f}, \boldsymbol{\theta}, \boldsymbol{\rho}) = \frac{1}{2} \|\mathcal{A}(\boldsymbol{\theta}, \boldsymbol{\rho})\mathbf{f} - \mathbf{g}\|^2_{\mathbb{C}^N}. \quad (1)$$

Alternating Optimization

Alternating Optimization exploits the separability of the unknowns $(\mathbf{f}, \boldsymbol{\theta}, \boldsymbol{\rho})$ and generates an iterative sequence $\{(\mathbf{f}^{(\ell)}, \boldsymbol{\theta}^{(\ell)}, \boldsymbol{\rho}^{(\ell)})\}$,

$$\mathbf{f}^{(\ell)} = \underset{\mathbf{f} \geq 0}{\operatorname{argmin}} \|\mathcal{A}(\boldsymbol{\theta}^{(\ell-1)}, \boldsymbol{\rho}^{(\ell-1)})\mathbf{f} - \mathbf{g}\|^2_{\mathbb{C}^N} \quad (2)$$

$$\boldsymbol{\theta}^{(\ell)} = \underset{\boldsymbol{\theta}^{min} \leq \boldsymbol{\theta} \leq \boldsymbol{\theta}^{max}}{\operatorname{argmin}} \|\mathcal{A}(\boldsymbol{\theta}, \boldsymbol{\rho}^{(\ell-1)})\mathbf{f}^{(\ell)} - \mathbf{g}\|^2_{\mathbb{C}^N} \quad (3)$$

$$\boldsymbol{\rho}^{(\ell)} = \underset{\boldsymbol{\rho}^{min} \leq \boldsymbol{\rho} \leq \boldsymbol{\rho}^{max}}{\operatorname{argmin}} \|\mathcal{A}(\boldsymbol{\theta}^{(\ell)}, \boldsymbol{\rho})\mathbf{f}^{(\ell)} - \mathbf{g}\|^2_{\mathbb{C}^N} \quad (4)$$

which converges to a stationary point for problem (1).
The subproblems (2), (3) and (4) can be solved **inexactly** [3] through a suitable descent method, such as Gradient Projection (GP) [1].

Alternating Optimization scheme

Choose $\mathbf{f}^{(0)} \geq 0$, $\boldsymbol{\theta}^{min} \leq \boldsymbol{\theta}^{(0)} \leq \boldsymbol{\theta}^{max}$, $\boldsymbol{\rho}^{min} \leq \boldsymbol{\rho}^{(0)} \leq \boldsymbol{\rho}^{max}$

FOR $\ell = 1, 2, \ldots$ DO:

STEP $\mathbf{f}$: Compute $\mathbf{f}^{(\ell)}$ by applying GP to (2) starting from $\mathbf{f}^{(\ell-1)}$.

STEP $\boldsymbol{\theta}$: Compute $\boldsymbol{\theta}^{(\ell)}$ by applying GP to (3) starting from $\boldsymbol{\theta}^{(\ell-1)}$.

STEP $\boldsymbol{\rho}$: Compute $\boldsymbol{\rho}^{(\ell)}$ by applying GP to (4) starting from $\boldsymbol{\rho}^{(\ell-1)}$.

END

Numerical results: the RHESSI mission

Figure 1: The RHESSI Spacecraft

- Launched by NASA on February 5, 2002
- **Goal**: study the particle acceleration mechanisms during solar flares
- Imaging system based on Rotating Modulation Collimators
- Provides estimates of a set of visibilities lying on 9 concentric circles in the Fourier plane, thus *the radial coordinate* $\boldsymbol{\rho}$ *is known*
- The other Fourier-based reconstruction algorithms assume $\boldsymbol{\theta}$ fixed

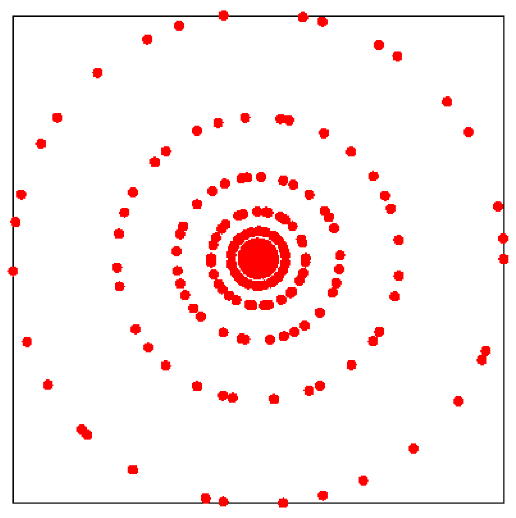

Figure 2: Sampling example in the frequency plane

Comparison between:

Deconvolution problem (Space-D) [2]:
$\boldsymbol{\theta} = \frac{1}{2}(\boldsymbol{\theta}^{min} + \boldsymbol{\theta}^{max})$, $\mathbf{f}$ unknown

Semi-blind deconvolution approach (AO):
both $\boldsymbol{\theta}$ and $\mathbf{f}$ unknowns

Considered datasets:

- **Sim1**: noise free
- **Sim2**: with realistic statistical noise
- **February 20, 2002**: real event (11:06:02–11:06:34 UT, energy band 22-26 keV)

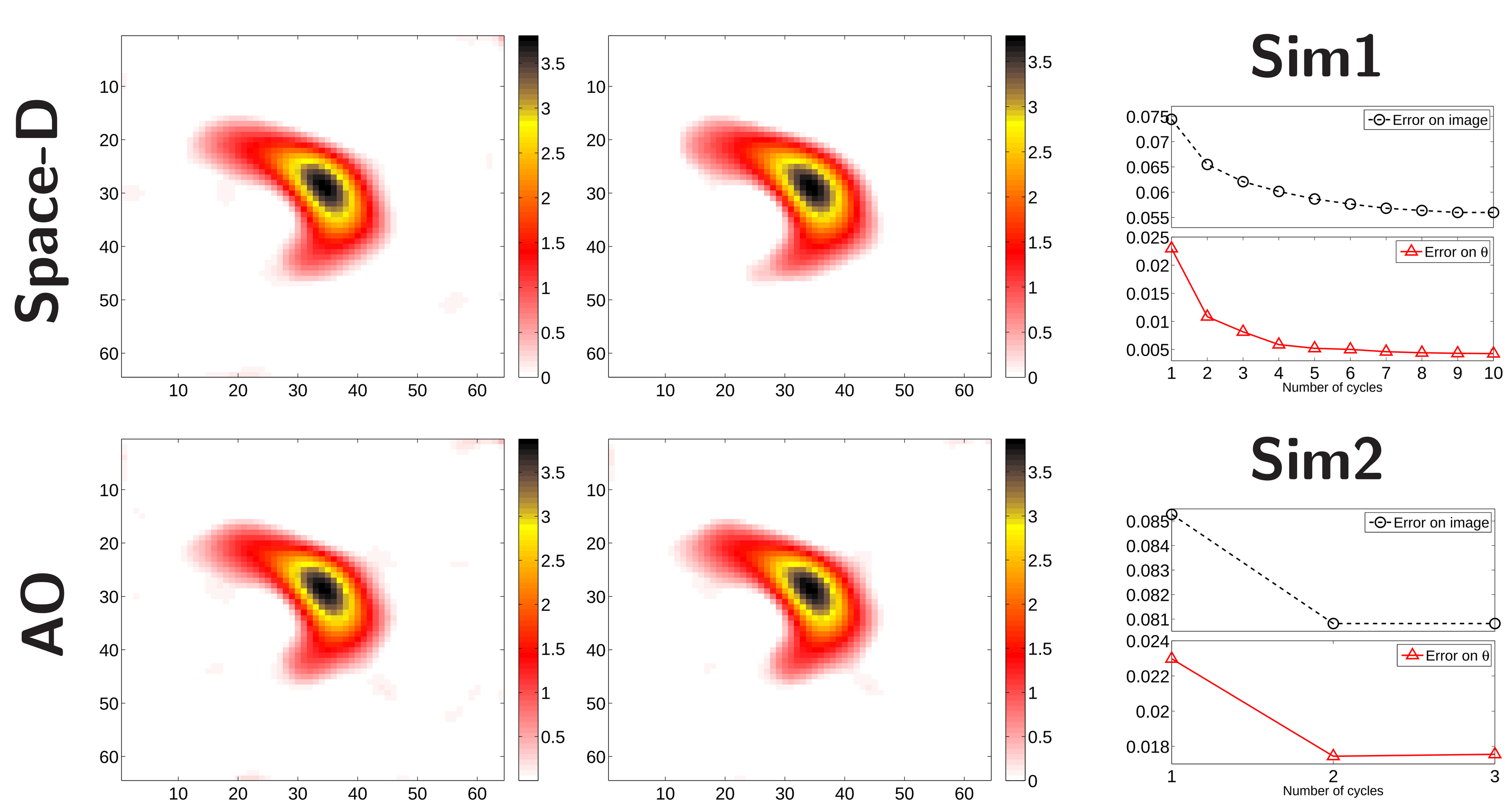

Figure 3: Reconstructed images and errors on $\mathbf{f}$ (black) and $\boldsymbol{\theta}$ (red)

February 20, 2002

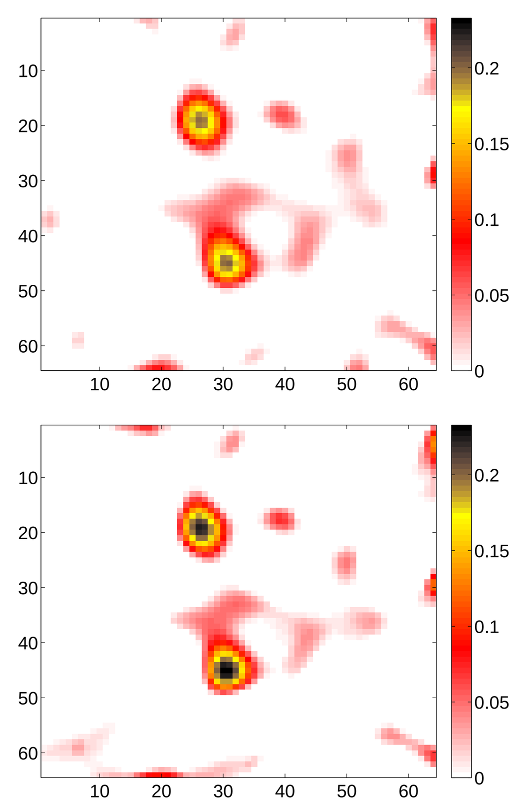

Figure 4: Real event

References

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