# Total Variation regularization algorithms for sparse tomography imaging from real data

E. Morotti†, E. Loli Piccolomini*

†University of Padova, *University of Bologna

## Introduction to 2d Computed Tomography

This work addresses the problem of 2d imaging reconstruction from few tomographic real data. Placing an object between an X-ray source and a detector, one of its slices is crossed by an X-ray fan-beam and we want to reconstruct its inner part. In modern Computed Tomography (CT) several projections are executed but it could be important to reduce the number of scans, avoiding to loose too much in quality. Here we compare how two different and efficient algorithms behave changing the sparisty of the views and working on a dataset obtained from a real tomographic experiment.

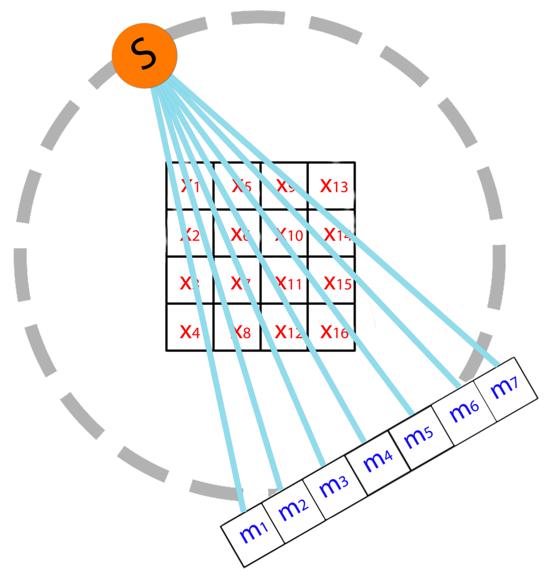


Figure: Scheme of a scan with an X-ray fan-beam.

In our case, the CT image formation process is modelled as an underdetermined linear system $Ax = b$ and the reconstructed image $x$ is the solution of the regularized problem:

$$\min_{x\in\mathbb{R}^N} f(x) = \frac{1}{2}\|Ax-b\|_2^2 + \lambda TV(x)$$

- $x$ is the slice of interest, discretized in $N := N_x \times N_y$ pixels;
- $b$ is the collection of all the noisy projection measurements, recorded by a linear detector (made of $n_{cells}$ elements) from $n_\theta$ angles taken on a circular trajectory (n $:= n_{cells} \times n_\theta$);
- $A$ is a matrix of size n$\times$N, representing how the projector works on the detector;
- $TV(x) := \sum_{j=1}^{N}\left(\|D_j x\|_2^2 + \beta^2\right)^{\frac{1}{2}}$ is the differentiable discrete Total Variation of $x$ (with $D_j x \in \mathbb{R}^2$ 2d-discrete gradient of pixel $x_j$) and $\lambda > 0$ is the regularization parameter.

Figure: Acquisition of actual projections for the lotus root phantom.

Due to the incompleteness of the projection data, n$\ll$N and the image reconstruction from real data becomes a challenging inverse problem.

## Scaled Gradient Projection (SGP) algorithm [1]

The constrained problem $\min_{x\geq 0} f(x)$ is solved with:

**SGP algorithm**

Initialize: $x_0 \geq 0,\ \delta,\sigma \in (0,1),\ 0 \leq \alpha_{min} \leq \alpha_0 \leq \alpha_{max},\ D_0 \in \mathcal{D}_\rho$;
while (convergence)
 $d_k = P_+(x_k - \alpha_k D_k \nabla f(x_k)) - x_k$; (scaled gradient projection step)
 $\lambda_k = 1$;
 while $f(x_k + \lambda_k d_k) > f(x_k) + \sigma\lambda_k \nabla f(x_k)^T d_k$
  $\lambda_k = \delta\lambda_k$; (backtracking step)
 end
 $x_{k+1} = x_k + \lambda_k d_k$;
 define the diagonal scaling matrix $D_{k+1} \in \mathcal{D}_\rho$;(scaling updating rule)
 define the step–length $\alpha_{k+1} \in [\alpha_{min}, \alpha_{max}]$;(step–length updating rule)
 k = k+1;
end

[1] R. Zanella et al., *Inverse Problems*, 2009

## Fixed Point (FP) algorithm [2]

Considering $L_k = L(x_k)$ where $L(x)$ is an operator such that $L(x_k)x_k = \nabla TV(x_k)$, the optimization problem is solved with:

**FP algorithm**

Initialize: $x_0$
while (convergence)
 $g_k = \nabla f(x_k)$;
 $H_k = A^t A + \lambda L(x_k)$; (Hessian approximation)
 $s_k$ such that $H_k s_k = -g_k$;
  (resolution of an inner inverse problem, with CG method)
 $x_{k+1} = x_k + s_k$;
 k=k+1;
end

[2] C. Vogel, *Computational methods for Inverse Problems*, SIAM 2002.

## Numerical results

**Problem parameters:**

- $N_x = N_y = 256$ (hence N = 65536)
- angular range = $[0, 2\pi]$
- $n_\theta = 120$ or $n_\theta = 20$
- $n_{cells} = 429$ (hence n = 51480 or n = 8580)
- data and image accuracies $\approx 280\ \mu m$

**Algorithm parameters:**

- $\lambda = 10^{-2}$ for SGP, $\lambda = 10^2$ for FP
- $\beta = 10^{-3}$
- maxiter_cg = 10

**Dataset:**[3]

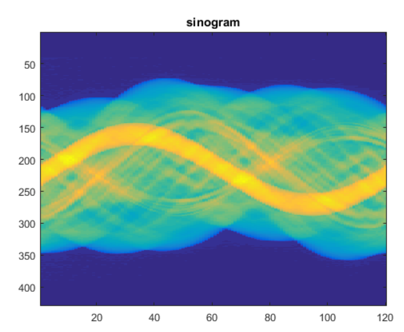
the object of interest is a central slice of a piece of lotus root filled with

- one circular chalk
- square pieces of ceramics
- a section of a pencil
- fragments of matches

[3] [http://www.fips.fi/dataset.php], [T. Bubba et al., *Tomographic X-ray data of a lotus root filled with attenuating objects*, arXiv:1609.07299, 2016]

### Results from 120 projections

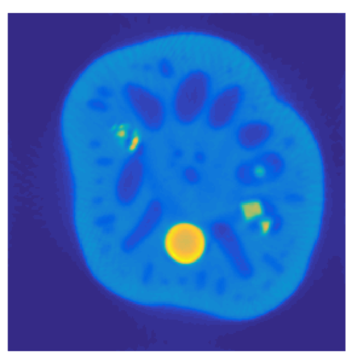
Sinogram:

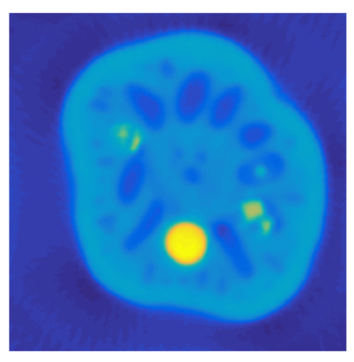


Reconstructions:

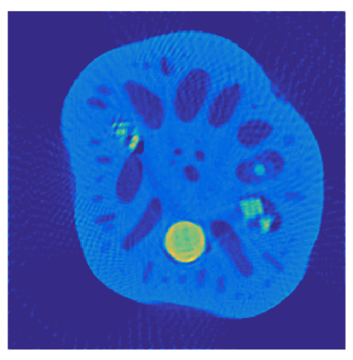
SGP FP

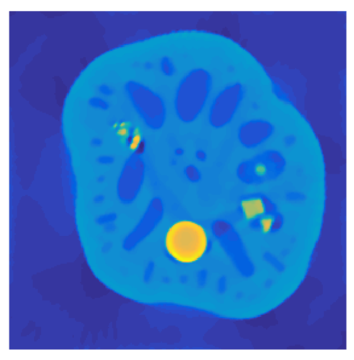
k=20





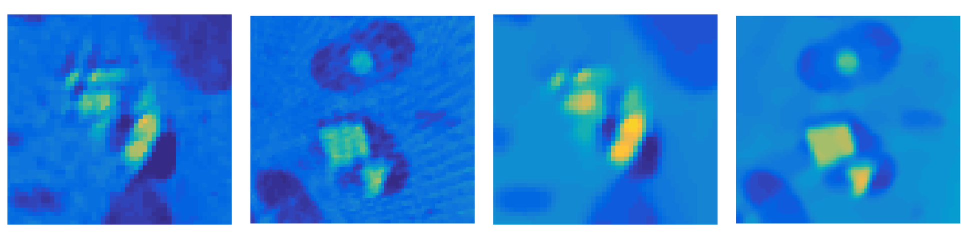
k=1000





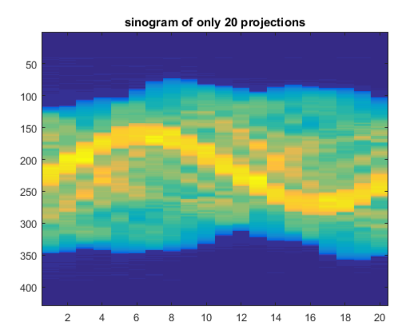
Zooms of reconstructions at k=1000:

SGP FP



### Results from 20 projections

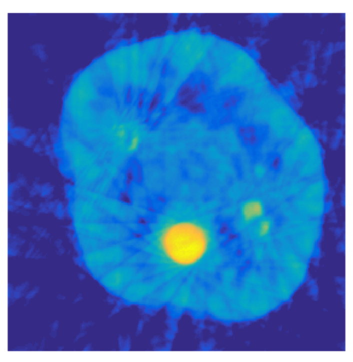
Sinogram:

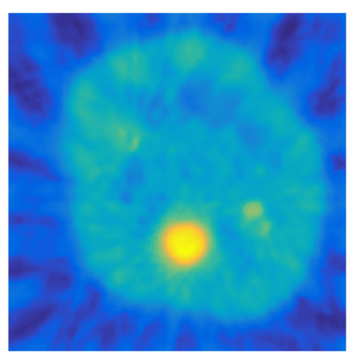


Reconstructions:

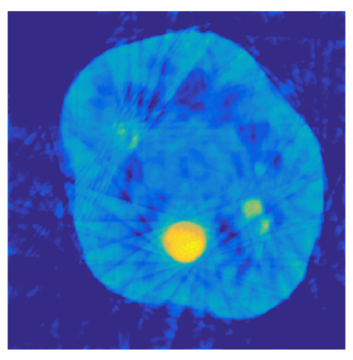
SGP FP

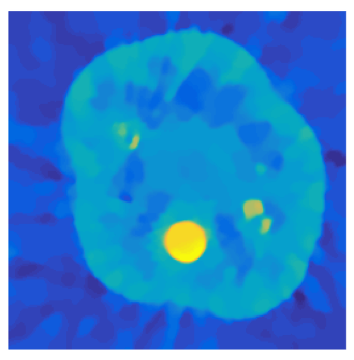
k=20





k=1000





Zooms of reconstructions at k=1000:

SGP FP